Select Solutions (Calculus on Surfaces)

(2d) Parametrize the portion of the sphere $x^2+y^2+z^2=4$ lying above the cone $z=\sqrt{x^2+y^2}$.

Pro-tip: Draw a picture!

Sol: The intersection of these surfaces (and thus the boundary of the desired surface) satisfies:

$$4 = x^2+y^2+z^2 = x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2 = 2(x^2+y^2)$$

$$\iff x^2+y^2 = 2.$$

Hence we can use polar coordinates to parametrize:

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$, so

$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), \sqrt{4-r^2}\rangle$ for $(r,\theta)\in[0,\sqrt{2}]\times[0,2\pi]$.

Alternatively we can use spherical coordinates. Note $\rho^2=4$ yields $\rho=2$ (reject negative root). Now to compute bounds on θ, note we need a full rotation of the circle to obtain all points on the surface.

Moreover, we compute the bounds for φ by noting that $z=\sqrt{x^2+y^2}$ has $\rho\cos(\varphi) = \sqrt{\rho^2\sin^2(\varphi)\left(\cos^2(\theta)+\sin^2(\theta)\right)} = \rho\sin(\varphi)$, so $\tan(\varphi)=1$ yields $\varphi=\frac{\pi}{4}$ on this cone. Hence:

$\vec{S}(\varphi,\theta) = \langle 2\cos(\theta)\sin(\varphi), 2\sin(\theta)\sin(\varphi), 2\cos(\varphi)\rangle$ for $(\varphi,\theta)\in[0,\frac{\pi}{4}]\times[0,2\pi]$

NB: There are (many!) other correct parametrizations... ////

(2e) Parametrize the portion of the plane $z = x+3$ lying inside the cylinder $x^2+y^2=1$.

Sol: Note that the region inside the cylinder is $x^2+y^2 \le 1$. We use polar coordinates and $z = x+3$ to parametrize:

$$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), r\cos(\theta)+3 \rangle \quad \text{on} \quad (r,\theta) \in [0,1] \times [0,2\pi].$$

NB: Many correct parametrizations exist... $\square$

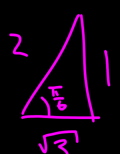
(3c) Compute a tangent plane to $\vec{S}(u,v) = \langle \sin(u), \cos(u)\sin(v), \sin(v) \rangle$ at the point where $(u,v) = (\frac{\pi}{6}, \frac{\pi}{6})$.

Sol: We recall planes have equation $\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$, where $\vec{n} = \vec{S}_u \times \vec{S}_v$ for parametrized surface $\vec{S}$.

$$\vec{S}_u = \langle \cos(u), -\sin(u)\sin(v), 0 \rangle$$

$$\vec{S}_v = \langle 0, \cos(u)\cos(v), \cos(v) \rangle$$

$$\therefore \vec{S}_u \times \vec{S}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(u) & -\sin(u)\sin(v) & 0 \\ 0 & \cos(u)\cos(v) & \cos(v) \end{vmatrix}$$



$$= \langle -\sin(u)\cos(v)\sin(v), -\cos(u)\cos(v), \cos^2(u)\cos(v) \rangle$$

$$\therefore \vec{n} = \left\langle -\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}, -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}, \left(\frac{\sqrt{3}}{2}\right)^2\cdot\frac{\sqrt{3}}{2} \right\rangle = -\frac{\sqrt{3}}{8}\langle 1, 2\sqrt{3}, -3 \rangle$$

and $\vec{x}_0 = \langle \frac{1}{2}, \frac{1}{2}\cdot\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2} \rangle$, so the desired tangent plane has equation

$$1\left(x - \frac{1}{2}\right) + 2\sqrt{3}\left(y - \frac{\sqrt{3}}{4}\right) - 3\left(z - \frac{\sqrt{3}}{2}\right) = 0$$

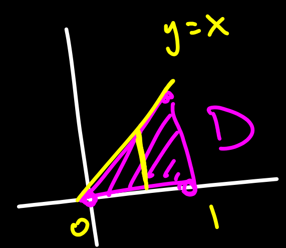
$\square$

(4d) Compute the area of the part of $z = 4 - x^2 + y$ lying above the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Sol: We have $\vec{S}(x,y) = \langle x, y, 4 - x^2 + y \rangle$ on

$$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$$

Picture!



Hence we compute the surface area as:

$$A = \iint_S 1\, dS = \iint_D |\vec{S}_x \times \vec{S}_y|\, dA.$$

Now compute

$\vec{S}_x = \langle 1, 0, -2x \rangle$ and $\vec{S}_y = \langle 0, 1, 1 \rangle$, so

$$\vec{S}_x \times \vec{S}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & 1 \end{vmatrix} = \langle 2x, -1, 1 \rangle$$ yields

$$|\vec{S}_x \times \vec{S}_y| = \sqrt{(2x)^2 + 1^2 + 1^2} = \sqrt{2 + 4x^2},$$ and thus:

$$A = \int_{x=0}^{1} \int_{y=0}^{x} \sqrt{2 + 4x^2}\, dy\, dx$$

$$= \int_{x=0}^{1} [y]_{y=0}^{x} \sqrt{2 + 4x^2}\, dx$$

$$= \frac{1}{8} \int_{x=0}^{1} 8x \sqrt{2 + 4x^2}\, dx$$

$u = 2 + 4x^2$ $\quad u(0) = 2$

$du = 8x\, dx$ $\quad u(1) = 6$

$$= \frac{1}{8} \int_{u=2}^{6} u^{1/2}\, du = \frac{1}{12} \left[u^{3/2} \right]_{u=2}^{6}$$

$$= \frac{1}{12} \left(6^{3/2} - 2^{3/2} \right) = \frac{1}{12} \left(2\sqrt{2} \left(3\sqrt{3} - 1 \right) \right) = \frac{3\sqrt{3} - 1}{3\sqrt{2}}$$ ////

↑ Good enough

(4f) Compute the area of the part of $y = x^2+z^2$ lying inside the cylinder $x^2+z^2=16$.

Sol: We parametrize the given surface by

$$\vec{S}(x,z) = \langle x,\ x^2+z^2,\ z\rangle \quad \text{on } D = \{(x,z) : x^2+z^2 \le 16\}.$$

More naturally using polar coordinates in the xz-plane:

$$\vec{S}(r,\theta) = \langle r\cos(\theta),\ r^2,\ r\sin(\theta)\rangle \quad \text{on } R = \{(r,\theta) : 0 \le r \le 4,\ 0 \le \theta \le 2\pi\}.$$

Now $\vec{S}_r = \langle \cos(\theta), 2r, \sin(\theta)\rangle$ and $\vec{S}_\theta = \langle -r\sin(\theta), 0, r\cos(\theta)\rangle$, so

$$\vec{S}_r \times \vec{S}_\theta = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & 2r & \sin(\theta) \\ -r\sin(\theta) & 0 & r\cos(\theta)\end{bmatrix} = \langle 2r^2\cos(\theta), -r, 2r^2\sin(\theta)\rangle.$$

↑ verify!

Thus $|\vec{S}_r \times \vec{S}_\theta| = \sqrt{4r^4\cos^2(\theta) + r^2 + 4r^4\sin^2(\theta)} = \sqrt{r^2(1+4r^2)} = r\sqrt{1+4r^2}$.

Finally we compute the desired surface area as:

$$A = \iint_S 1\, dS = \iint_R |\vec{S}_r \times \vec{S}_\theta|\, dA$$

$$= \int_{r=0}^{4}\int_{\theta=0}^{2\pi} r\sqrt{1+4r^2}\, d\theta\, dr \qquad \left(\begin{array}{ll} u = 1+4r^2 & u(4) = 65 \\ du = 8r\,dr & u(0) = 1 \end{array}\right.$$

$$= \int_{r=0}^{4} [\theta]_{\theta=0}^{2\pi}\, r\sqrt{1+4r^2}\, dr = \frac{2\pi}{8}\int_{r=0}^{4}\sqrt{1+4r^2}\, 8r\, dr$$

$$= \frac{\pi}{4}\int_{u=1}^{65} u^{1/2}\, du = \frac{\pi}{6}\left[u^{3/2}\right]_{u=1}^{65} = \frac{\pi}{6}\left(65\sqrt{65}-1\right)$$

▨